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AN EQUILIBRIUM FINITE ELEMENT METHOD FOR FOURTH ORDER ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract—An equilibrium finite element method for 4th order elliptic problems with variable coefficients on convex polygonal domains has been developed. In the particular case of bending problems of elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness, this new equilibrium method allows a simultaneous approximation to the displacement and the bending and twisting moment tensor. Error estimates for equilibrium finite element solution are given.

1. INTRODUCTION

In Refs [1–5] four different mixed finite element methods have been developed for the Dirichlet problem of 4th order elliptic partial differential equations with variable/constant coefficients, the biharmonic problem and the bending problems of elastic–anisotropic/orthotropic/isotropic plates with variable/constant thickness being particular cases of this problem, for which the mixed methods of [1–3] give a simultaneous approximation to displacement u, curvature tensor $(u_{,ij})$ and bending and twisting moment tensor (ψ_{ij}), and those of [4] and [5] allow a simultaneous approximation to displacement u and bending and twisting moment tensor (ψ_{ij}). In the mixed method of [5], the continuity of the normal bending moment M_n at the interelement boundaries is required as in the mixed method of Hellan–Hermann–Johnson for the biharmonic problem [6–13]. In [14] Fraeijs de Veubeke developed the equilibrium finite element method in which the continuity of both normal bending moment M_n and Kirchhoff transverse force K_n is required at the interelement boundaries, [15] being a subsequent paper based on the general procedure developed in [14]. Hence, the equilibrium finite element method may be regarded as a natural extension of the Hellan–Hermann–Johnson mixed method for the biharmonic problem [7, 13, 16].

The present paper is devoted to the development of such an equilibrium finite element method for the Dirichlet problem of 4th order partial differential equations with variable/constant coefficients. This paper contains new results in this direction which were first announced in the research report [17]. In fact, for bending problems of anisotropic/orthotropic/isotropic plates with variable/constant thickness, this new equilibrium finite element method gives also a simultaneous approximation to displacement u and bending and twisting moment tensor (ψ_{ij}). Moreover, the equilibrium method [13, 16] for the biharmonic problem, which allows a simultaneous approximation to displacement u, change in curvature tensor $(u_{,ij})$ (but not the "actual" bending and twisting moment tensor (ψ_{ij}), i.e. further computation will be necessary to find bending and twisting moments), can be retrieved as a particular case from the general scheme developed in this paper by means of a suitable choice of the coefficients of the equation. Moreover, it was *not* known [16] whether for $k > 3$ the set $(\Sigma 1)$ (see (5.35)) represents the degrees of freedom of tensor-valued functions $\Phi_h = (\phi_{hij}) \in \mathbf{V}_h$ (see (5.5)), although it has been claimed in [16] that for $k \leqslant 3$, this set $(\Sigma 1)$ does represent the degrees of freedom of $\Phi_h \in \mathbf{V}_h$ and the corresponding discrete Babuska–Brezzi condition (i.e. assumption A3) holds for $k \leqslant 3$. But this paper contains a very interesting result that for $k \geqslant 6$, $(\Sigma 1)$ does *not* represent degrees of freedom of $\Phi_h \in \mathbf{V}_h$ and consequently, the corresponding discrete Babuska–Brezzi condition (i.e. the assumption A3) does *not* hold for $k \geqslant 6$. Finally, error estimates for the equilibrium finite element solution have been developed.

2. NOTATION

Let Ω be a convex polygon with boundary Γ in $\mathbb{R}^2$ and $\mathbf{H}^m(\Omega)$ be the Sobolev spaces [18, 19] of integral order $m \geqslant 0$ equipped with inner product $\langle .,. \rangle_{m,\Omega}$, norm $\|.\|_{m,\Omega}$ and seminorm $|.|_{m,\Omega}$ such that $H^0(\Omega) = L^2(\Omega)$,

$$H_0^1(\Omega) = \{v : v \in H^1(\Omega), \gamma_0 v = v|_\Gamma = 0\},$$

$$H_0^2(\Omega) = \{v : v \in H^2(\Omega), \gamma_0 v = v|_\Gamma = 0, \gamma_1 v = (\partial v/\partial n)|_\Gamma = 0\}, \tag{2.1}$$

where $\partial v/\partial n$ is the derivative of v in the direction of the exterior normal to Γ; γ_k: $H^m(\Omega) \to H^{m-k-1/2}(\Gamma)$ are trace operators [18, 19], $m = 1, 2$; $k = 0, m-1$; $H^{1/2}(\Gamma)$, $H^{3/2}(\Gamma)$ are the fractional order Sobolev spaces of functions on Γ; $H_0^m(\Omega) \equiv \overline{D(\Omega)}$ in the norm topology of $H^m(\Omega)$, $m \geqslant 0$, $D(\Omega)$ being the space of test functions on Ω.

For $p > 2$, let $W_0^{1,p}(\Omega)$ be the Sobolev space [18, 19] defined by

$$W_0^{1,p}(\Omega) = \{v : v \in W^{1,p}(\Omega), \gamma_0 v = v|_\Gamma = 0\},$$

such that $\forall p > 2$,

$$H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega), \tag{2.2}$$

with dense, continuous injections, and $\forall p > 2$,

$$H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega}), \tag{2.3}$$

(2.2) and (2.3) being the consequences of the Sobolev's imbedding theorem [18–20].

3. THE CONTINUOUS VARIATIONAL PROBLEM

To the Dirichlet problem (P) defined by: for given $f \in L^2(\Omega)$, find u such that

$$\Lambda u = f \text{ in } \Omega, \quad u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0, \tag{3.1}$$

where

$$(\Lambda u)(x) = \frac{\partial^2}{\partial x_k \partial x_l}\left(a_{ijkl}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)(x) \equiv (a_{ijkl}u_{,ij})_{,kl}(x) \quad \text{in} \quad \Omega, \tag{3.2}$$

(in (3.2) and also in the sequel, the Einstein's summation convention has been followed), we associate the following Galerkin variational problem ($\mathrm{P_G}$) defined by: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = l(v), \quad \forall v \in H_0^2(\Omega), \tag{3.3}$$

where the continuous, symmetric, bilinear form $a(.,.)$ and the continuous linear form $l(.)$ are defined by

$$a(v, w) = \langle \Lambda v, w \rangle_{0,\Omega} = \int_\Omega a_{ijkl} v_{,ij} w_{,kl} \, d\Omega, \quad \forall v, w \in H_0^2(\Omega); \tag{3.4}$$

$$l(v) = \langle f, v \rangle_{0,\Omega} = \int_\Omega f v \, d\Omega, \quad \forall v \in H_0^2(\Omega); \tag{3.5}$$

the coefficients a_{ijkl} satisfy the following conditions: $\forall i, j, k\, l = 1, 2$,

(A1): $a_{ijkl} \in C^\circ(\bar{\Omega})$, $a_{ijkl} \geqslant 0$, $a_{ijkl}(x) = a_{klij}(x)$, $\forall x \in \bar{\Omega}$; but without loss of generality [21], we can always assume that $\forall i, j, k, l = 1, 2$;

(A1′): $a_{ijkl}(x) = a_{klij}(x) = a_{ijlk}(x) = a_{jilk}(x)$, $\forall x \in \bar{\Omega}$; and in the sequel we will assume that (A1′) holds;

(A2′): $\exists \alpha > 0$ such that $a(v, v) \geqslant \alpha \|v\|_{2,\Omega}^2$, $\forall v \in H_0^2(\Omega)$;

(A2′): $\forall \xi = (\xi_{11}, \xi_{22}, \xi_{12}, \xi_{21}) \in \mathbb{R}^4$, with $\xi_{12} = \xi_{21}$, $\exists \alpha_0 > 0$ such that

$$a_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant \alpha_0 \|\xi\|_{\mathbb{R}^4}^2, \quad \forall x \in \bar{\Omega}.$$

Remark 3.1

In [21], sufficient conditions for (A2′) to hold can be found along with interesting examples, and it has also been shown that (A2′)$\Rightarrow$(A2).

Theorem 3.1 [21]

Under the assumptions (A1)–(A2′), the problem (P_G) has a unique solution.

4. EQUILIBRIUM METHOD FORMULATION

For the construction of suitable bilinear forms of the equilibrium method to be formulated, we prepare the following results. To every $\xi = (\xi_{11}, \xi_{22}, \xi_{12}, \xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, we associate $\bar{\xi} = (\xi_{11}, \xi_{22}, \xi_{12}) \in \mathbb{R}^3$ such that $\forall x \in \bar{\Omega}$,

$$a_{ijkl}(x)\xi_{ij}\xi_{kl} = \bar{\xi}[A(x)]\bar{\xi}^{\mathrm{T}} \geqslant \alpha_0 \|\xi\|^2_{\mathbb{R}^4} \geqslant \alpha_0 \|\bar{\xi}\|^2_{\mathbb{R}^3},$$

where a_{ijkl} satisfy (A1)–(A2′); $\bar{\xi}^{\mathrm{T}}$ is the transpose of $\bar{\xi}$; $[A(x)] \in \mathscr{L}(\mathbb{R}^3)$ is defined by:

$$[A(x)] = \begin{bmatrix} a_{1111}(x) & a_{1122}(x) & 2a_{1112}(x) \\ a_{2211}(x) & a_{2222}(x) & 2a_{2212}(x) \\ 2a_{1211}(x) & 2a_{1222}(x) & 4a_{1212}(x) \end{bmatrix} = [A(x)]^{\mathrm{T}}, \quad \forall x \in \bar{\Omega}. \tag{4.1}$$

Proposition 4.1

$\forall x \in \bar{\Omega}$, $[A(x)] \in \mathscr{L}(\mathbb{R}^3)$, defined by (4.1) is symmetric, positive-definite, and its inverse $[A^{-1}(x)] \in \mathscr{L}(\mathbb{R}^3)$ defined by: $\forall x \in \bar{\Omega}$, $[A(x)][A^{-1}(x)] = I$,

$$[A^{-1}(x)] = \begin{bmatrix} A_{1111}(x) & A_{1122}(x) & A_{1112}(x) \\ A_{2211}(x) & A_{2222}(x) & A_{2212}(x) \\ A_{1211}(x) & A_{1222}(x) & A_{1212}(x) \end{bmatrix} = [A^{-1}(x)]^{\mathrm{T}} \tag{4.2}$$

where $I \in \mathscr{L}(\mathbb{R}^3)$ is the identity matrix; for $1 \leqslant i \leqslant j \leqslant 2$, $1 \leqslant k \leqslant l \leqslant 2$, $A_{ijkl} = A_{klij}$, is also symmetric, positive-definite.

Proposition 4.2 [17]

$\forall x \in \bar{\Omega}$, the symmetric $[A^*(x)] \in \mathscr{L}(\mathbb{R}^3)$ defined by:

$$[A^*(x)] = \begin{bmatrix} A_{1111}(x) & A_{1122}(x) & 2A_{1112}(x) \\ A_{2211}(x) & A_{2222}(x) & 2A_{2212}(x) \\ 2A_{1211}(x) & 2A_{1222}(x) & 4A_{1212}(x) \end{bmatrix} = [A^*(x)]^{\mathrm{T}} \tag{4.3}$$

is positive-definite, i.e.

$$\forall x \in \bar{\Omega}, \quad \forall \bar{\xi} = (\xi_{11}, \xi_{22}, \xi_{12}) \in \mathbb{R}^3, \quad \exists \alpha_1 > 0 \tag{4.4}$$

such that $\bar{\xi}[A^*(x)]\bar{\xi}^{\mathrm{T}} \geqslant \alpha_1 \|\bar{\xi}\|^2_{\mathbb{R}^3}$.

Now, define new functions A_{2121}, A_{ij21}, A_{21ij} $(1 \leqslant i \leqslant j \leqslant 2)$ with the help of functions A_{ijkl} in (4.2) as follows: $A_{2121} = A_{1212}$, $A_{ij21} = A_{21ij} = A_{ij12}$, $1 \leqslant i \leqslant j \leqslant 2$, from which, together with (4.2), we have: $\forall x \in \bar{\Omega}$,

$$A_{ijkl}(x) = A_{klij}(x) = A_{lkij}(x) = A_{lkji}(x), \quad \forall i, j, k, l = 1, 2. \tag{4.5}$$

Proposition 4.3 [17]

$\forall x \in \bar{\Omega}$, $\forall \xi = (\xi_{11}, \xi_{22}, \xi_{12}, \xi_{21}) \in \mathbb{R}^4$ with $\xi_{12} = \xi_{21}$, $\forall \zeta = (\zeta_{11}, \zeta_{22}, \zeta_{12}, \zeta_{21}) \in \mathbb{R}^4$ with $\zeta_{12} = \zeta_{21}$,

$$A_{ijkl}(x)a_{ijmn}(x)\xi_{mn}\zeta_{kl} = \xi_{ij}\zeta_{ij}; \tag{4.6}$$

$$A_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant \alpha_2 \|\xi\|^2_{\mathbb{R}^4} \quad \text{with some} \quad \alpha_2 > 0; \tag{4.7}$$

for

$$i = 1, 2, \quad A_{iikl}(x)a_{iikl}(x) = 1, \quad 2A_{12kl}(x)a_{12kl}(x) = 1, \tag{4.8}$$

$A_{ijkl}(x)a_{mnkl}(x) = 0$ for $i \neq m$ or $j \neq m$, $1 \leqslant i \leqslant j \leqslant 2$, $1 \leqslant m \leqslant n \leqslant 2$.

Remark 4.1

$\forall i, j, k, l = 1, 2$, the functions $A_{ijkl} \in C^0(\bar{\Omega})$, which follows from (A1) and the Proposition 4.1.

Remark 4.2

From (A1), $a_{ijkl}(x) \geqslant 0 \; \forall x \in \bar{\Omega}$, $\forall i, j, k, l = 1, 2$, but A_{ijkl} are not positive-valued functions in general, i.e. there may exist some $i, j, k, l = 1, 2$ and some $x \in \bar{\Omega}$ such that $A_{ijkl}(x) < 0$.

Now, we define the following spaces:

$$\left.\begin{aligned} &\mathbf{H} = \{\Phi : \Phi = (\phi_{ij}), \quad 1 \leqslant i, j \leqslant 2, \quad \phi_{ij} \in L^2(\Omega), \quad \phi_{12} = \phi_{21}\} \\ &\text{with} \\ &\|\Phi\|^2_{\mathbf{H}} = \|\Phi\|^2_{0,\Omega} = \sum_{i,j=1}^{2} \|\phi_{ij}\|^2_{0,\Omega}, \quad \forall \Phi \in \mathbf{H}; \end{aligned}\right\} \tag{4.9}$$

$$W = W_0^{1,p}(\Omega), \quad p > 2; \quad \|\chi\|_W = \|\chi\|_{1,p,\Omega}, \quad \forall \chi \in W. \tag{4.10}$$

Let T_h be an admissible triangulation [20] of $\bar{\Omega}$ into closed triangles T. Given a triangle $T \in T_h$ with vertices $\{a_{i,T}\}_{i=1}^3$ and boundary ∂T and a tensor-valued function $\Phi = (\phi_{ij})_{i,j=1,2}$ with $\phi_{ij} \in H^2(T)$ and $\phi_{ij} = \phi_{ji}$, $1 \leqslant i, j \leqslant 2$, we define the "normal bending moment" $M_n(\Phi)$, "twisting moment" $M_{nt}(\Phi)$, "transverse shear force" $Q_n(\Phi)$, "Kirchhoff transverse force" $K_n(\Phi)$, corresponding to the bending moment tensor field Φ along ∂T as follows:

$$M_n(\Phi) = \phi_{ij} n_i n_j; \quad M_{nt}(\Phi) = \phi_{ij} n_i t_j; \quad Q_n(\Phi) = \phi_{ij,i} n_j; \quad K_n(\Phi) = \frac{\partial M_{nt}(\Phi)}{\partial t} + Q_n(\Phi), \tag{4.11}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit exterior normal and $\mathbf{t} = (t_1, t_2) = (n_2, -n_1)$ is the unit tangent along ∂T. Then, we have the following Green's formulae:

$$\forall \phi_{ij} \in H^2(T), \quad 1 \leqslant i, j \leqslant 2, \quad \phi_{12} = \phi_{21}, \quad \forall \chi \in H^2(T),$$

$$\int_T \phi_{ij} \chi_{,ij} \, \mathrm{d}T = -\int_T \phi_{ij,j} \chi_{,i} \, \mathrm{d}T + \int_{\partial T} \left(M_n(\Phi) \frac{\partial \chi}{\partial n} + M_{nt}(\Phi) \frac{\partial \chi}{\partial t} \right) \mathrm{d}s; \tag{4.12}$$

$$= \int_T \phi_{ij,ij} \chi \, \mathrm{d}T + \int_{\partial T} \left(M_n(\Phi) \frac{\partial \chi}{\partial n} + M_{nt}(\Phi) \frac{\partial \chi}{\partial t} - Q_n(\Phi) \chi \right) \mathrm{d}s; \tag{4.13}$$

$$= \int_T \phi_{ij,ij} \chi \, \mathrm{d}T + \int_{\partial T} \left(M_n(\Phi) \frac{\partial \chi}{\partial n} - K_n(\Phi) \chi \right) \mathrm{d}s + \sum_{i=1}^{3} J_{i,T}(\Phi) \chi(a_{i,T}), \tag{4.14}$$

where

$$J_{i,T}(\Phi) = M_{nt}(\Phi)(a^+_{i,T}) - M_{nt}(\Phi)(a^-_{i,T}) = \text{jump of } M_{nt}(\Phi) \text{ at the vertex } a_{i,T}, \quad 1 \leqslant i \leqslant 3. \tag{4.15}$$

Definition

Let $\Phi \in \mathbf{H}$ with $\phi_{ij}|_T \in H^2(T) \; \forall T \in T_h$, $1 \leqslant i, j \leqslant 2$. Then, $M_n(\Phi)$ (resp. $K_n(\Phi)$) defined in (4.11) is said to be "continuous at the interelement boundaries" of the triangulation T_h, if and only if for any pair T_1, T_2 of adjacent triangles of T_h with a common side $T_1 \cap T_2$,

$$M_{n_1}(\Phi|_{T_1}) = M_{n_2}(\Phi|_{T_2}) \quad (\text{resp. } K_{n_1}(\Phi|_{T_1}) = -K_{n_2}(\Phi|_{T_2})) \quad \text{on} \quad T_1 \cap T_2 = \partial T_1 \cap \partial T_2, \tag{4.16}$$

where $\mathbf{n}_i = (n_1^i, n_2^i)$ is the unit exterior normal to the boundary ∂T_i of T_i, $i = 1, 2$.

Now, we can define the admissible space of moment fields of the equilibrium method formulation as follows:

$$\left.\begin{aligned} \mathbf{V} = \{\Phi : \Phi \in \mathbf{H}, \phi_{ij}|_T \in H^2(T), \quad &\forall T \in T_h, 1 \leqslant i, j \leqslant 2, \\ &M_n(\Phi) \text{ and } K_n \text{ are continuous at the interelement boundaries}\}, \\ \|\Phi\|^2_{\mathbf{V}} = \sum_{i,j=1}^{2} &\sum_{T \in T_h} \|\phi_{ij}\|^2_{2,T}. \end{aligned}\right\} \tag{4.17}$$

Obviously, we have $\mathbf{V} \hookrightarrow \mathbf{H}$, $W \hookrightarrow H_0^1(\Omega)$ with continuous imbeddings such that

$$\forall \Phi \in \mathbf{V}, \quad \|\Phi\|_{\mathbf{H}} \leqslant \|\Phi\|_{\mathbf{V}}; \quad \forall \chi \in W, \quad \|\chi\|_{1,\Omega} \leqslant \sigma_0 \|\chi\|_W \quad \text{for some} \quad \sigma_0 > 0. \tag{4.18}$$

Proposition 4.4 [17]

(i) $\Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2}, \quad \phi_{12} = \phi_{21}, \quad \phi_{ij} \in H^2(\Omega), \quad \forall i,j = 1,2 \Rightarrow \Phi \in \mathbf{V}.$

(ii) $$\forall \Phi = (\phi_{ij}) \in \mathbf{V}, \quad \forall \chi \in H_0^2(\Omega) \int_\Omega \phi_{ij}\chi_{,ij}\, \mathrm{d}\Omega = \sum_{T \in T_h} \int_T \phi_{ij,ij}\chi \, \mathrm{d}T + \sum_{a_{i,T} \in N_{oh}} J_{i,T}(\Phi)\chi(a_{i,T}), \tag{4.19}$$

where N_h (resp. N_{oh}) denote the set of all vertices (resp. all interior vertices) of triangles of T_h in $\bar{\Omega}$ (resp. Ω).

Now we define the continuous bilinear forms: $A(.,.)$: $\mathbf{H} \times \mathbf{H} \to \mathbb{R}$; $b(.,.)$: $\mathbf{V} \times W \to \mathbb{R}$ as follows:

$$A(\Psi, \Phi) = \int_\Omega A_{ijkl}\psi_{ij}\phi_{kl}\, \mathrm{d}\Omega = A(\Phi, \Psi), \quad \forall \Psi = (\psi_{ij}), \quad \Phi = (\phi_{ij}) \in \mathbf{H}, \tag{4.20}$$

where $A_{ijkl} = A_{ijkl}(x)$ are defined in (4.1)–(4.8);

$$b(\Phi, \chi) = -\sum_{T \in T_h} \left[\int_T \phi_{ij,ij}\chi \, \mathrm{d}T + \sum_{i=1}^{3} J_{i,T}(\Phi)\chi(a_{i,T}) \right], \tag{4.21}$$

where $J_{i,T}(\Phi)$ are defined in (4.15).

Proposition 4.5

$A(.,.)$ and $b(.,.)$ defined by (4.20) and (4.21) are continuous on $\mathbf{H} \times \mathbf{H}$ and $\mathbf{V} \times W$ respectively, i.e. $\exists$ constants $M > 0$, $m^* > 0$ such that

$$|A(\Psi, \Phi)| \leqslant M \|\Psi\|_{0,\Omega}\|\Phi\|_{0,\Omega}, \quad \forall \Psi, \Phi \in \mathbf{H}, \tag{4.22}$$

$$|b(\Phi, \chi)| \leqslant m^* \|\Phi\|_{\mathbf{V}}\|\chi\|_W, \qquad \forall \Phi \in \mathbf{V}, \quad \forall \chi \in W. \tag{4.23}$$

Theorem 4.1

(i) $A(.,.)$ defined by (4,20) is $\mathbf{H}$-elliptic, i.e.

$$\exists \alpha_2 > 0 \quad \text{such that} \quad \forall \Phi \in \mathbf{H}, \quad A(\Phi, \Phi) \geqslant \alpha_2 \|\Phi\|_{0,\Omega}^2; \tag{4.24}$$

(ii) $$\exists \beta > 0 \quad \text{such that} \quad \forall \chi \in W, \quad \sup_{\Phi \in \mathbf{V}} \frac{b(\Phi, \chi)}{\|\Phi\|_{\mathbf{V}}} \geqslant \beta \|\chi\|_{0,\Omega}. \tag{4.25}$$

Proof.

(i) From (4.20) and (4.7), we have

$$A(\Phi, \Phi) = \int_\Omega A_{ijkl}(x)\phi_{ij}(x)\phi_{kl}(x)\, \mathrm{d}\Omega \geqslant \int_\Omega \alpha_2 \phi_{ij}(x)\phi_{ij}(x)\, \mathrm{d}\Omega = \alpha_2 \|\Phi\|_{0,\Omega}^2.$$

(ii) The proof is given in [13] and [17].

Now, for the problem (P_G) defined in (3.3)–(3.5), we can construct the problem (Q) of the equilibrium method under consideration as follows: find $(\Psi, \lambda) \in \mathbf{V} \times W$ such that

$$(Q): \quad A(\Psi, \Phi) + b(\Phi, \lambda) = 0, \quad \forall \Phi \in \mathbf{V}, \tag{4.26}$$

$$b(\Psi, \chi) = -\langle f, \chi \rangle_{0,\Omega}, \quad \forall \chi \in W, \tag{4.27}$$

where $A(.,.)$ and $b(.,.)$ are defined by (4.20) and (4.21) respectively.

Theorem 4.2 [17]

The problem (Q) has at most one solution.

Since $A(.,.)$ is not *a priori* $\mathbf{V}$-elliptic, the problem (Q) is not well-posed in general, i.e. the existence of solution of (Q) cannot be proved in general. But we have

Theorem 4.3

If u be the solution of the problem (P_G) such that $u \in H_0^2(\Omega) \cap H^4(\Omega)$ and $a_{ijkl}u_{,kl} \in H^2(\Omega)$, $\forall i, j = 1, 2$, then $(\Psi, \lambda) = (\Psi, u) \in \mathbf{V} \times W$ with $\Psi = (\psi_{ij})_{1 \leqslant i,j \leqslant 2}$, $\psi_{ij} = a_{ijkl}\lambda_{,kl}$ is the solution of the problem (Q).

Conversely, if $(\Psi, \lambda) \in \mathbf{V} \times W$ be the solution of the problem (Q) with $\Psi = (\psi_{ij})$, $1 \leqslant i, j \leqslant 2$, then $\lambda = u \in H_0^2(\Omega)$ is the solution of the problem (P_G) and

$$\psi_{ij} = a_{ijkl}\lambda_{,kl} = a_{ijkl}u_{,kl}, \quad \forall i, j = 1, 2. \tag{4.28}$$

Proof. Let $u \in H_0^2(\Omega) \cap H^4(\Omega)$ be the solution of (P_G) with $a_{ijkl}u_{,kl} \in H^2(\Omega)$, $\forall i, j = 1, 2$, i.e.

$$\int_\Omega a_{ijkl}u_{,ij}v_{,kl}\,\mathrm{d}\Omega = \langle f, v\rangle_{0,\Omega}, \quad \forall v \in H_0^2(\Omega). \tag{4.29}$$

Set $\lambda = u$; $\psi_{ij} = a_{ijkl}u_{,kl} = a_{ijkl}\lambda_{,kl} \in H^2(\Omega)$. Then, from Proposition 4.4, $\Psi \in \mathbf{V}$. Now, for $\lambda \in H_0^2(\Omega)$, $\forall \Phi \in \mathbf{V}$,

$$b(\Phi, \lambda) = -\sum_{T \in T_h}\left[\int_T \phi_{ij,ij}\lambda\,\mathrm{d}T + \sum_{i=1}^{3} J_{i,T}(\Phi)\lambda(a_{i,T})\right] = -\int_\Omega \phi_{ij}\lambda_{,ij}\,\mathrm{d}\Omega \quad \text{(from (4.19))}.$$

Hence, for $(\Psi, \lambda) \in \mathbf{V} \times W$ and $\forall \Phi \in \mathbf{V}$, we have

$$\begin{aligned} A(\Psi, \Phi) + b(\Phi, \lambda) &= \int_\Omega A_{ijkl}(x)\psi_{ij}(x)\phi_{kl}(x)\,\mathrm{d}\Omega - \int_\Omega \phi_{ij}(x)\lambda_{,ij}(x)\,\mathrm{d}\Omega \\ &= \int_\Omega A_{ijkl}(x)a_{ijmn}(x)\lambda_{,mn}(x)\phi_{kl}(x)\,\mathrm{d}\Omega - \int_\Omega \phi_{ij}(x)\lambda_{,ij}(x)\,\mathrm{d}\Omega \\ &= \int_\Omega \lambda_{,ij}(x)\phi_{ij}(x)\,\mathrm{d}\Omega - \int_\Omega \lambda_{,ij}(x)\phi_{ij}(x)\,\mathrm{d}\Omega \equiv 0 \quad \text{(from (4.6))}. \end{aligned}$$

Thus, $(\Psi, \lambda) \in \mathbf{V} \times W$ satisfies (4.26). Again, we have: $\forall \chi \in H_0^2(\Omega)$,

$$b(\Psi, \chi) = -\int_\Omega \psi_{ij}\chi_{,ij}\,\mathrm{d}\Omega = -\int_\Omega a_{ijkl}u_{,kl}\chi_{,ij}\,\mathrm{d}\Omega = -\langle f, \chi\rangle_{0,\Omega} \quad \text{(from (4.29))}$$

$\Rightarrow$ (4.27) holds for $\Psi \in \mathbf{V}$ and $\forall \chi \in H_0^2(\Omega)$.

Since $H_0^2(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, $p > 2$, (by virtue of 2.2) and $b(\Psi, .)$ is continuous on W, (4.27) will also hold for all $\chi \in W_0^{1,p}(\Omega)$, $p > 2$, i.e. $(\Psi, \lambda) \in \mathbf{V} \times W$ satisfies (4.27). Consequently, $(\Psi, \lambda) \in \mathbf{V} \times W$ is a solution of the problem (Q) and its uniqueness follows from the Theorem 4.2.

Conversely, let $(\Psi, \lambda) \in \mathbf{V} \times W$ be the solution of the problem (Q) with $\Psi = (\psi_{ij})$, $1 \leqslant i, j \leqslant 2$. Define $\Phi^* = (\phi\delta_{ij})_{1 \leqslant i,j \leqslant 2}$ with $\phi \in D(\Omega)$. Then, $\Phi^* \in \mathbf{V}$. Since $(\Psi, \lambda) \in \mathbf{V} \times W$ is the solution of (Q), we have from (4.26):

$$\begin{aligned} &\forall \phi \in D(\Omega), \quad A(\Psi, \Phi^*) + b(\Phi^*, \lambda) = 0 \\ &\Rightarrow \int_\Omega A_{ijkl}\psi_{ij}(\phi\delta_{kl})\,\mathrm{d}\Omega - \int_\Omega (\phi\delta_{kl})_{,kl}\lambda\,\mathrm{d}\Omega \\ &= 0, \quad \forall \phi \in D(\Omega) \\ &\Rightarrow \int_\Omega (A_{ijkl}\psi_{ij}\delta_{kl})\phi\,\mathrm{d}\Omega \\ &= -\int_\Omega \lambda_{,i}\phi_{,i}\,\mathrm{d}\Omega, \quad \forall \phi \in H_0^1(\Omega), \end{aligned}$$

since $D(\Omega)$ is dense in $H_0^1(\Omega)$.

Since $A_{ijkl}\psi_{ij}\delta_{kl} \in L^2(\Omega)$, Ω is a convex polygon, $\lambda \in H_0^1(\Omega) \cap H^2(\Omega)$ [22] with

$$\Delta\lambda = A_{ijkl}\psi_{ij}\delta_{kl}. \tag{4.30}$$

Again, choosing $\tilde{\Phi} = (\tilde{\phi}\delta_{ij})_{1 \leqslant i,j \leqslant 2}$ with $\tilde{\phi} \in D(\bar{\Omega})$, we have $\tilde{\Phi} \in \mathbf{V}$, $\forall \tilde{\phi} \in D(\bar{\Omega})$. Then, for $\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$, $\forall \tilde{\phi} \in D(\bar{\Omega})$, we have

$$b(\tilde{\Phi}\ \lambda) = -\sum_{T\in T_h}\int_T (\tilde{\phi}\delta_{ij})_{,ij}\lambda\ \mathrm{d}T = -\int_\Omega (\Delta\tilde{\phi})\lambda\ \mathrm{d}\Omega$$

$$= \int_\Omega \tilde{\phi}_{,i}\lambda_{,i}\,\mathrm{d}\Omega = -A(\Psi, \tilde{\Phi})$$

$$\Rightarrow \forall\tilde{\phi}\in H^1(\Omega),\quad \int_\Omega \lambda_{,i}\tilde{\phi}_{,i}\,\mathrm{d}\Omega = -\int_\Omega (A_{ijkl}\psi_{ij}\delta_{kl})\tilde{\phi}\ \mathrm{d}\Omega$$

$$\Rightarrow \forall\tilde{\phi}\in H^1(\Omega),\quad -\int_\Omega (\Delta\lambda)\tilde{\phi}\ \mathrm{d}\Omega + \int_\Gamma \frac{\partial\lambda}{\partial n}\tilde{\phi}\ \mathrm{d}\Gamma = -\int_\Omega (\Delta\lambda)\tilde{\phi}\ \mathrm{d}\Omega),\quad \text{(from (4.30))},$$

$$\Rightarrow \langle\gamma_1\lambda, \gamma_0\tilde{\phi}\rangle_{0,\Gamma} = 0,\quad \forall\tilde{\phi}\in H^1(\Omega)$$

$$\Rightarrow \gamma_1\lambda = (\partial\lambda/\partial n)|_\Gamma = 0,$$

since $\gamma_1\lambda$, $\gamma_0\tilde{\phi}\in H^{1/2}(\Gamma)$, and $H^{1/2}(\Gamma) = \gamma_0(H^1(\Omega))$.

Then, $\lambda\in H_0^1(\Omega)\cap H^2(\Omega)$ with $\gamma_1\lambda = 0 \Rightarrow \lambda\in H_0^2(\Omega)$. Now, for $(\Psi, \lambda)\in \mathbf{V}\times W$ with $\lambda\in H_0^2(\Omega)$, we are to prove (4.28), i.e. $\psi_{ij} = a_{ijkl}\lambda_{,kl}$, $\forall i, j = 1, 2$. For this, we have from (4.26): for $(\Psi, \lambda)\in\mathbf{V}\times W$ with $\lambda\in H_0^2(\Omega)$, $\forall\Phi = (\phi_{ij})\in\mathbf{V}$,

$$\int_\Omega A_{ijkl}\psi_{ij}\phi_{kl}\,\mathrm{d}\Omega = \int_\Omega \phi_{ij}\lambda_{,ij}\,\mathrm{d}\Omega \qquad \text{(by 4.19)}$$

$$\Rightarrow \int_\Omega (A_{ijkl}\psi_{ij} - \lambda_{,kl})\phi_{kl}\,\mathrm{d}\Omega = 0,\quad \forall\Phi\in\mathbf{V}.$$

Choose $\Phi = (\phi, 0, 0, 0)$, $\tilde{\Phi} = (0, \phi, \phi, 0)$, and $\underset{\sim}{\Phi} = (0, 0, 0, \phi)$ with $\phi\in D(\Omega)$. Then $\Phi, \tilde{\Phi}\ \underset{\sim}{\Phi}\in\mathbf{V}$ and we get:

$$\forall\phi\in D(\Omega),\quad \int_\Omega (A_{ij11}\psi_{ij} - \lambda_{,11})\phi\ \mathrm{d}\Omega = 0,$$

$$\int_\Omega (A_{ij22}\psi_{ij} - \lambda_{,22})\ \phi\ \mathrm{d}\Omega = 0,$$

$$\int_\Omega [(A_{ij12}\psi_{ij} - \lambda_{,12}) + (A_{ij21}\psi_{ij} - \lambda_{,21})]\phi\ \mathrm{d}\Omega = 0$$

$$\Leftrightarrow \forall k, l = 1, 2,\quad A_{ijkl}\psi_{ij} - \lambda_{,kl} = 0,$$

since $\forall k, l = 1, 2$, $A_{ijkl}\psi_{ij} - \lambda_{,kl}\in L^2(\Omega)$ and $D(\Omega)$ is dense in

$$L^2(\Omega) \Rightarrow \lambda_{,kl} = A_{ijkl}\psi_{ij},\quad k, l = 1, 2 \tag{4.31}$$

$$\Rightarrow \forall m, n = 1, 2,\quad a_{mnkl}\lambda_{,kl} = a_{mnkl}A_{ijkl}\psi_{ij} = \psi_{mn}$$

by virtue of (4.8), and (4.28) is thus established.

It remains to show that $\lambda\in H_0^2(\Omega)$ satisfying (4.28) is the solution of the problem ($\mathrm{P_G}$). For this, we have from (4.27), (4.28) and (4.19):

$$\forall\chi\in H_0^2(\Omega),\quad b(\Psi, \chi) = -\int_\Omega \psi_{ij}\chi_{,ij}\,\mathrm{d}\Omega = -\langle f, \chi\rangle_{0,\Omega} \Rightarrow \int_\Omega a_{ijkl}\lambda_{,kl}\chi_{,ij}\,\mathrm{d}\Omega = \langle f, \chi\rangle_{0,\Omega},\quad \forall\chi\in H_0^2(\Omega)$$

with

$$\lambda\in H_0^2(\Omega) \Rightarrow \lambda = u\in H_0^2(\Omega)$$

is the solution of the problem ($\mathrm{P_G}$) by the Theorem 3.1.

Examples

Here, we shall consider only the biharmonic problem and bending problems of elastic anisotropic/orthotropic/isotropic plates with variable/constant thickness.

I. The biharmonic problem

Choosing $a_{ijkl}=\delta_{ik}\delta_{jl}$, we get the biharmonic operator $\Lambda \equiv \Delta\Delta$ and the corresponding bilinear form $A(.,.)$ defined by:

$$A(\Psi,\Phi)=\int_\Omega \delta_{ik}\delta_{jl}\psi_{ij}\phi_{kl}\,\mathrm{d}\Omega=\int_\Omega \psi_{ij}\phi_{ij}\,\mathrm{d}\Omega \tag{4.32}$$

which is $H_0^2(\Omega)$-elliptic [21], and (A2′) holds with $\alpha_0=1$. Then, the solution $(\Psi,\lambda)\in \mathbf{V}\times W$ of the corresponding mixed problem (Q):

$$\int_\Omega \psi_{ij}\phi_{ij}\,\mathrm{d}\Omega-\sum_{T\in T_h}\left[\int_T \phi_{ij,ij}\lambda\,\mathrm{d}T+\sum_{i=1}^{3}J_{i,T}(\Phi)\lambda(a_{i,T})\right]=0,\qquad \forall\Phi\in\mathbf{V},$$

$$-\sum_{T\in T_h}\left[\int_T \psi_{ij,ij}\chi\,\mathrm{d}T+\sum_{i=1}^{3}J_{i,T}(\Psi)\chi(a_{i,T})\right]=-\langle f,\chi\rangle_{0,\Omega},\quad \forall\chi\in W,$$

which is the equilibrium formulation for the biharmonic equation given in [13] and [16], is characterized by $\lambda=u$, $\Psi=(\psi_{ij})$ with $\psi_{ij}=\delta_{ik}\delta_{jl}u_{,kl}=u_{,ij}$, $\forall i,j=1,2$, where $u\in H_0^2(\Omega)\cap H^4(\Omega)$ is the solution of the associated problem $(\mathrm{P_G})$.

Remark 4.3

For $a_{ijkl}=\delta_{ik}\delta_{jl}$, (A1) holds, but *not* (A1′), since $a_{ijkl}=a_{klij}$, $a_{ijkl}\neq a_{jikl}$ and $a_{ijkl}\neq a_{ijlk}$ in general. Following [21], we can define new coefficients $\bar{a}_{ijkl}=(a_{ijkl}+a_{jikl}+a_{jilk}+a_{ijlk})/4$ for which (A1′) holds, subsequent steps remaining unchanged.

Remark 4.4

The bilinear form (4.32) corresponds to that for the bending problem of isotropic plate with constant thickness, when Poisson's coefficient $\nu=0$, bending rigidity $D=1$ [see isotropic case (iii)]. Then, $\lambda=u$ is the deflection of the bent plate and $\kappa=$ change in curvature tensor $=$ bending moment tensor Ψ, but for $\nu\neq 0$, $D\neq 1$, κ is *not* the bending moment tensor.

II. Bending problems of elastic plates with variable/constant thickness

In all the following examples, the thickness function h satisfies the following condition:

$$h\in C^0(\bar{\Omega}),\quad h_0=\min_{(x_1,x_2)\in\bar{\Omega}}h(x_1,x_2)>0\quad\text{such that}\quad h\in W^{2,\infty}(\Omega)\ [2]. \tag{4.33}$$

(i) For anisotropic case [21, 23]

$$\left.\begin{aligned}&a_{iiii}=D_{ii},\quad a_{1212}=a_{1221}=a_{2112}=a_{2121}=D_{66},\quad a_{1122}=a_{2211}=D_{12},\\&a_{1112}=a_{1121}=a_{1211}=a_{2111}=D_{16},\quad a_{1222}=a_{2122}=a_{2212}=a_{2221}=D_{26},\end{aligned}\right\} \tag{4.34}$$

where $D_{ij}=B_{ij}h^3/12$ denote rigidities [23] having the properties:

$$\left.\begin{aligned}&D_{11},D_{22},D_{66}>0,\quad D_{12}=\nu_1D_{22}=\nu_2D_{11},\quad 0\leqslant\nu_i<1/2\quad(i=1,2),\\&0\leqslant D_{16}<(1-\nu_2)D_{11},\quad 0\leqslant D_{26}<(1-\nu_1)D_{22},\quad D_{16}+D_{26}<D_{66},\end{aligned}\right\} \tag{4.35}$$

B_{ij}s being the expression [23] given in terms of the elastic constants of the generalized Hooke's law for the anisotropic material of the plate, h satisfies (4.33) such that $\psi_{ij}=a_{ijkl}u_{,kl}\in H^2(\Omega)$ [2], $\forall i,j=1,2$, when the solution u of $(\mathrm{P_G})$ belongs to $H^4(\Omega)\cap H_0^2(\Omega)$. (A1)–(A2′) hold [21] and $a(.,.)$ is $H_0^2(\Omega)$-elliptic [21]. Then, the coefficients A_{ijkl} are determined from (4.1)–(4.8). Thus, from (4.1)–(4.3), we have:

$$\begin{aligned}&A_{iiii}=4(D_{jj}D_{66}-D_{j6}^2)/|A(.)|\quad(i\neq j);\\&A_{1212}=(D_{11}D_{22}-D_{12}^2)/|A(.)|;\\&A_{1112}=2(D_{12}D_{26}-D_{16}D_{22})/|A(.)|;\\&A_{1122}=4(D_{16}D_{26}-D_{12}D_{66})/|A(.)|;\\&A_{2212}=2(D_{16}D_{12}-D_{11}D_{26})/|A(.)|,\end{aligned} \tag{4.36}$$

where

$$|A(x)| = \det A(x) = 4[D_{11}(D_{22}D_{66} - D_{26}^2) + D_{12}(D_{16}D_{26} - D_{12}D_{66}) + D_{16}(D_{12}D_{26} - D_{16}D_{22})](x), \quad \forall x \in \bar{\Omega}, \tag{4.37}$$

and then, all other A_{ijkl} are determined using (4.5) and the corresponding bilinear form $A(.,.)$ in (Q) is given by: $\forall \Psi, \Phi \in \mathbf{V}$,

$$\begin{aligned} A(\Psi, \Phi) = \int_\Omega \frac{4}{|A(x)|} [\{(D_{22}D_{66} - D_{26}^2)\psi_{11} + (D_{16}D_{26} - D_{12}D_{66})\psi_{22} \\ + (D_{12}D_{26} - D_{16}D_{22})\psi_{12}\}\phi_{11} + \{(D_{16}D_{26} - D_{12}D_{66})\psi_{11} \\ + (D_{11}D_{66} - D_{16}^2)\psi_{22} + (D_{16}D_{12} - D_{11}D_{26})\psi_{12}\}\phi_{22} \\ + \{(D_{12}D_{26} - D_{16}D_{22})\psi_{11} + (D_{16}D_{12} - D_{11}D_{26})\psi_{22} \\ + (D_{11}D_{22} - D_{12}^2)\psi_{12}\}\phi_{12}] \, d\Omega, \end{aligned} \tag{4.38}$$

$b(.,.)$ being defined by (4.21).

The solution $(\Psi, \lambda) \in \mathbf{V} \times W$ of (Q) is characterized by: $\lambda = u$ is the deflection of the bent plate, $\Psi = (\psi_{ij})$ is the "actual" bending moment tensor with bending moments ψ_{ii} in the x_i-direction $(i = 1, 2)$ and twisting moment $\psi_{12} = \psi_{21}$, i.e. one obtains directly and simultaneously "u" and ψ_{ij}s.

(ii) The orthotropic case [21, 23–26] can be obtained from the anisotropic case (i) by putting in (4.35)–(4.38),

$$\begin{aligned} D_{ii} &= D_i, \\ D_{12} &= \nu_1 D_2 = \nu_2 D_1, \\ D_{66} &= D_t, \\ D_{16} &= D_{26} = 0 \end{aligned}$$

with

$$\begin{aligned} D_i &= E_i h^3/(12(1 - \nu_1\nu_2)), \\ D_t &= Gh^3/12, \\ G &= E_1E_2/(E_1 + (1 + 2\nu_1)E_2), \\ E_1\nu_2 &= E_2\nu_1, \end{aligned} \tag{4.39}$$

E_i, ν_i being Young's moduli and Poisson's coefficients $(i = 1, 2)$.

Then, the corresponding bilinear form $A(.,.)$ is given by:

$$A(\Psi, \Phi) = \int_\Omega \left[\frac{1}{D_1(1 - \nu_1\nu_2)}(\psi_{11} - \nu_1\psi_{22})\phi_{11} + \frac{1}{D_2(1 - \nu_1\nu_2)}(-\nu_2\psi_{11} + \psi_{22})\phi_{22} + \frac{1}{D_t}\psi_{12}\phi_{12}\right] d\Omega, \tag{4.40}$$

and the solution $(\Psi, \lambda) \in \mathbf{V} \times W$ is characterized by the deflection λ and "actual" bending moment tensor $\Psi = (\psi_{ij})$ as in the anisotropic case (i).

(iii) The results of the isotropic case are obtained from the orthotropic case (ii) by putting in (4.39)–(4.40) $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = \nu$ and consequently,

$$D_1 = D_2 = D = Eh^3/(12(1 - \nu^2)) \tag{4.41}$$

with the solution $(\Psi, \lambda) \in \mathbf{V} \times W$ still characterized by: $\lambda = u$ is the deflection, $\Psi = (\psi_{ij})$ is the "actual" bending moment tensor.

Remark 4.5

The equilibrium method of [13] and [16] for the isotropic case (iii) gives simultaneously the deflection u and the change in curvature tensor $\kappa = (u_{,ij})$, but *not* the "actual" bending moment tensor $\Psi = (\psi_{ij})$ for which additional computations will be necessary using the formulae:

$$\psi_{11} = D(u_{,11} + \nu u_{,22}), \quad \psi_{22} = D(\nu u_{,11} + u_{,22}), \quad \psi_{12} = D(1-\nu)u_{,12}. \tag{4.42}$$

Remark 4.6

If the "change in curvature" tensor $(u_{,ij})$ is to be determined then it can be easily done by using the formula (4.31), i.e.

$$\forall\, i, j = 1, 2, \quad u_{,ij} = \lambda_{,ij} = A_{ijkl}\psi_{kl}$$

with A_{ijkl} defined by (4.1)–(4.8).

5. FINITE ELEMENT APPROXIMATION

Let $\{\lambda_i\}$ be the barycentric coordinates with respect to the vertices $\{a_{i,T}\}$ of a triangle $T \in T_h$, ∂T being the boundary of T, T_h being the admissible triangulation introduced earlier. Let $P_k(T)$ be the linear space of restrictions to T of all polynomials in x_1 and x_2 and of degree $\leqslant k$. Then, for $k \geqslant 2$ define the set of all "bubble functions" of $P_{k+1}(T)$, i.e.

$$B_{k+1}(T) = \{q : q \in P_{k+1}(T), q\,|_{\partial T} = 0\} = \{q : q = \lambda_1\lambda_2\lambda_3 p, p \in P_{k-2}(T)\} \tag{5.1}$$

is the set of all bubble functions of $P_{k+1}(T)$.

Remark 5.1

If we would have let $k = 0, 1$ in the definition of $B_{k+1}(T)$, then $B_{k+1}(T) = \emptyset$ for $k = 0, 1$.

Proposition 5.1

For $k \geqslant 2$, $B_{k+1}(T)$ defined by (5.1) is a linear space with $\dim B_{k+1}(T) = \dim P_{k-2}(T)$.

For $k \geqslant 1$ we define the space of polynomials

$$P^*_{k+1}(T) = P_1(T) + B_{k+1}(T) \tag{5.2}$$

such that for $k = 1$,

$$P^*_2(T) = P_1(T), \tag{5.3}$$

for $k = 2$,

$$P^*_3(T) = P_1(T) + \{\lambda_1\lambda_2\lambda_3\alpha\} \quad \text{with} \quad \alpha \in \mathbb{R}. \tag{5.4}$$

Now, we introduce the following finite dimensional spaces: for $k \geqslant 1$,

$$\mathbf{V}_h = (\Phi_h : \Phi_h \in \mathbf{V}, \Phi_h = (\phi_{hij})_{,l \leqslant 1, j \leqslant 2}, \phi_{hij}\,|_T \in P_k(T), \forall\, T \in T_h\} \tag{5.5}$$

$$W_h = \{\chi_h : \chi_h \in C^0(\bar{\Omega}), \chi_h\,|_T \in P^*_{k+1}(T), \forall\, T \in T_h, \chi_h\,|_\Gamma = 0\} \tag{5.6}$$

such that

$$\mathbf{V}_h \subset \mathbf{V} \subset \mathbf{H}, \quad W_h \subset W \subset H^1_0(\Omega). \tag{5.7}$$

Proposition 5.2

(i) For $k = 1$,

$$\dim W_h = \text{total number of interior vertices in } \Omega. \tag{5.8}$$

(ii) For $k \geqslant 2$,

$$\dim W_h = \text{total number of interior vertices in } \Omega + (\dim P_{k-2}) \times \text{number of triangles in } \bar{\Omega}. \tag{5.9}$$

Now, we can construct the equilibrium finite element problem $(\mathrm{Q_h})$ corresponding to the continuous problem (Q) as follows: find $(\Psi_h, \lambda_h) \in \mathbf{V}_h \times W_h$ such that

$$(\mathrm{Q}): \quad A(\Psi_h, \Phi_h) + b(\Phi_h, \lambda_h) = 0, \quad \forall\, \Phi_h \in \mathbf{V}_h, \tag{5.10}$$

$$b(\Psi_h, \chi_h) = -\langle f, \chi_h \rangle_{0,\Omega}, \quad \forall\, \chi_h \in W_h, \tag{5.11}$$

where $A(.,.)$ and $b(.,.)$ are defined by (4.20) and (4.21) respectively. We make the following important assumption (external ellipticity condition or discrete Babuska–Brezzi condition [8]):

$$\text{(A3):}\quad \exists \beta_1 > 0 \quad \text{such that} \quad \sup_{\Phi_h \in \mathbf{V}_h} \frac{|b(\Phi_h, \chi_h)|}{\|\Phi_h\|_{\mathrm{V}}} \geqslant \beta_1 \|\chi_h\|_{0,\Omega}, \quad \forall \chi_h \in W_h.$$

Remark 5.2

It will be shown later that for a specific choice of degrees of freedom for tensor-valued functions in $\mathbf{V}_h$ (see Proposition 5.3), the assumption (A3) holds (see Proposition 5.4).

Theorem 5.1 [17]

Under the assumption (A3), the equilibrium finite element problem $(\mathrm{Q_h})$ has a unique solution.

Since the existence and uniqueness of the solution of the problem $(\mathrm{Q_h})$ have been established under the assumption (A3), now we shall show that (A3) holds, if we introduce the degrees of freedom for functions in $\mathbf{V}_h$ in a specific manner.

Let S_h be the set of sides in the triangulation T_h such that $T_i^* \in S_h$ $(1 \leqslant i \leqslant 3)$ are the three sides of a triangle $T \in T_h$. $\forall T \in T_h$ define the linear space $\mathbf{V}_T$ as follows:

$$\mathbf{V}_T = \{\Phi_h : \Phi_h = (\phi_{hij})_{1 \leqslant i,j \leqslant 2},\, \phi_{h12} = \phi_{h21},\, \phi_{hij} \in P_k(T), \quad \forall i, j = 1, 2\}, \tag{5.12}$$

where $P_k(T)$ is the linear space of restrictions to $T \in T_h$ of all polynomials of degree $\leqslant k$ in variables x_1 and x_2, $k \geqslant 1$.

Then,

$$\dim \mathbf{V}_T = 3(k+2)(k+1)/2. \tag{5.13}$$

Consider the set Σ_T of linearly independent linear functionals on $\mathbf{V}_T$ defined by: for $\Phi_h = (\phi_{hij}) \in \mathbf{V}_T$,

$$\Sigma_T = \left\{ \int_{T_i^*} M_n(\Phi_h) q \, \mathrm{d}s,\, q \in P(T_i^*),\, 1 \leqslant i \leqslant 3;\, \int_{T_i^*} K_n(\Phi_h) q \, \mathrm{d}s,\, q \in P_{k-1}(T_i^*),\, 1 \leqslant i \leqslant 3; \right.$$

$$\left. \int_T \phi_{hij} b_{,ij} p \, \mathrm{d}T,\, b \in B_{k+1}(T),\, p \in P_1(T) \right\}, \tag{5.14}$$

$$\operatorname{card}(\Sigma_T) = 3(k+1) + 3k + 3k(k-1)/2 = 3(k+2)(k+1)/2 = \dim \mathbf{V}_T. \tag{5.15}$$

Lemma 5.1

For $1 \leqslant k \leqslant 3$, Σ_T defined by (5.14) is $\mathbf{V}_T$-unisolvent.

Proof. For $1 \leqslant k \leqslant 3$, the proof of the $\mathbf{V}_T$-unisolvence of Σ_T can be found in [16].

In [16] it has also been stated that for $k > 3$, there is *no* proof for the $\mathbf{V}_T$-unisolvence of Σ_T. But recently the following result has been obtained:

Lemma 5.2

For $k \geqslant 6$, Σ_T defined by (5.14) is *not* $\mathbf{V}_T$-unisolvent.

Proof. It is sufficient to show that for $k \geqslant 6$, $\exists \Phi_h^* = (\phi_{hij}^*) \neq 0$ in $\mathbf{V}_T$ such that

$$\int_{T_i^*} M_n(\Phi_h^*) q \, \mathrm{d}s = 0, \quad q \in P_k(T_i^*), \quad 1 \leqslant i \leqslant 3; \tag{5.16}$$

$$\int_{T_i^*} K_n(\Phi_h^*) q \, \mathrm{d}s = 0, \quad q \in P_{k-1}(T_i^*), \quad 1 \leqslant i \leqslant 3; \tag{5.17}$$

$$\int_T \phi_{hij}^* b_{,ij} p \, \mathrm{d}T = 0, \quad b \in B_{k+1}(T), \quad p \in P_1(T). \tag{5.18}$$

The proof is given in three steps.

Step 1. For $k \geqslant 6$, the following auxiliary set Σ_T^* of linearly independent linear functionals on $\mathbf{V}_T$ defined by: $\forall \Phi_h = (\phi_{hij}^*) \in \mathbf{V}_T$,

$$\Sigma_T^* = \left\{ \int_{T_i^*} M_n(\Phi_h^*) q \, \mathrm{d}s, \, q \in P_k(T_i^*), \, 1 \leqslant i \leqslant 3; \right.$$

$$\int_{T_i^*} K_n(\Phi_h^*) q \, \mathrm{d}s, \, q \in P_{k-1}(T_i^*), \, 1 \leqslant i \leqslant 3;$$

$$\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*) b \, \mathrm{d}T, \, b \in B_{k+2}(T),$$

$$\left. \int_T (\phi_{h12,1}^* + \phi_{h22,2}^*) b \, \mathrm{d}T, \, b \in B_{k+2}(T) \right\} \quad \text{is not } \mathbf{V}_T\text{-unisolvent.} \tag{5.19}$$

Since for $k \geqslant 6$,

$$\operatorname{card}(\Sigma_T^*) = 3(k+1) + 3k + \frac{k(k+1)}{2} + \frac{k(k+1)}{2}$$

$$= 3(k+1) + 3k + k(k+1) < 3(k+1) + 3k + 3(k-1)k/2 = \dim \mathbf{V}_T, \tag{5.20}$$

Σ_T^* defined by (5.19) is *not* $\mathbf{V}_T$-unisolvent, and hence, for $k \geqslant 6$, there exists a *nonzero* $\Phi_h^* = (\phi_{hij}^*) \in \mathbf{V}_T$ (i.e. $\exists \Phi_h^* = (\phi_{hij}^*) \neq 0$ in $\mathbf{V}_T$) such that

$$\int_{T_i^*} M_n(\Phi_h^*) q \, \mathrm{d}s = 0, \quad q \in P_k(T_i^*), \quad 1 \leqslant i \leqslant 3, \tag{5.21}$$

$$\int_{T_i^*} K_n(\Phi_h^*) q \, \mathrm{d}s = 0, \quad q \in P_{k-1}(T_i^*), \quad 1 \leqslant i \leqslant 3, \tag{5.22}$$

$$\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*) b \, \mathrm{d}T = 0, \quad b \in B_{k+2}(T), \tag{5.23}$$

$$\int_T (\phi_{h12,1}^* + \phi_{h22,2}^*) b \, \mathrm{d}T = 0, \quad b \in B_{k+2}(T). \tag{5.24}$$

Step 2. For $k \geqslant 6$, for any nonzero $\Phi_h^* = (\phi_{hij}^*) \in \mathbf{V}_T$ satisfying (5.21)–(5.24)

$$M_n(\Phi_h^*) = 0 \quad \text{on} \quad \partial T, \tag{5.25}$$

$$\phi_{hij,ij}^* = 0 \quad \text{in} \quad T. \tag{5.26}$$

Since $M_n(\Phi_h^*|_{T_i^*}) \in P_k(T_i^*)$, $1 \leqslant i \leqslant 3$, from (5.21), the result (5.25) immediately follows, if we choose $q = M_n(\Phi_h^*|_{T_i^*})$, $1 \leqslant i \leqslant 3$.

Now, we prove (5.26),

$$\Phi_h^* = (\phi_{hij}^*) \in \mathbf{V}_T \Rightarrow (\phi_{h11,1}^* + \phi_{h21,2}^*) \in P_{k-1}(T).$$

Then, choosing

$$b = \lambda_1 \lambda_2 \lambda_3 (\phi_{h11,1}^* + \phi_{h21,2}^*) \in B_{k+2}(T),$$

we get from (5.23):

$$\int_T (\phi_{h11,1}^* + \phi_{h21,2}^*)^2 \lambda_1 \lambda_2 \lambda_3 \, \mathrm{d}T = 0 \Leftrightarrow \phi_{h11,1}^* + \phi_{h21,2}^* = 0 \quad \text{in} \quad T, \tag{5.27}$$

since $\lambda_i > 0$ in $\mathring{T}$, $1 \leqslant i \leqslant 3$ $\mathring{T} = \operatorname{int}(T)$.

Similarly, we get from (5.24):

$$\phi_{h12,1}^* + \phi_{h22,2}^* = 0 \quad \text{in} \quad T. \tag{5.28}$$

Now, differentiating both sides of (5.27) and (5.28) with respect to x_1 and x_2 respectively and then adding, we get

$$\phi^*_{hij,ij} = 0 \quad \text{in} \quad T. \tag{5.29}$$

Step 3. For $k \geqslant 6$, any nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24) will also satisfy (5.16)–(5.18), i.e. for $k \geqslant 6$, Σ_T defined by (5.14) is not $\mathbf{V}_T$-unisolvent.

For $k \geqslant 6$, a nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24) satisfies (5.16) and (5.17). So, it remains to show that such a nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ will also satisfy (5.18). For this, we write the left hand side of (5.18) as follows:

$$\text{for} \quad b \in B_{k+1}(T), \quad \bar{\alpha}_i \in \mathbb{R}, \quad i = 0, 1, 2, \quad \int_T \phi^*_{hij} b_{,ij} (\bar{\alpha}_0 + \bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2)\, \mathrm{d}T$$

$$= \bar{\alpha}_0 \int_T \phi^*_{hij} b_{,ij}\, \mathrm{d}T + \bar{\alpha}_1 \int_T \phi^*_{hij} b_{,ij} x_1\, \mathrm{d}T + \bar{\alpha}_2 \int_T \phi^*_{hij} b_{,ij} x_2\, \mathrm{d}T. \tag{5.30}$$

Now, for $k \geqslant 6$, if each of the three integrals on the right hand side of (5.30) vanishes, then for $k \geqslant 6$, any nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24) will also satisfy (5.18). Hence, it is sufficient to prove that for $k \geqslant 6$, for nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24), we have:

$$\int_T \phi^*_{hij} b_{,ij}\, \mathrm{d}T = 0, \quad b \in \mathrm{B}_{k+1}(T), \tag{5.31}$$

$$\int_T \phi^*_{hij} b_{,ij} x_1\, \mathrm{d}T = 0, \quad b \in B_{k+1}(T), \tag{5.32}$$

$$\int_T \phi^*_{hij} b_{,ij} x_2\, \mathrm{d}T = 0, \quad b \in B_{k+1}(T). \tag{5.33}$$

From the Green's formula (Corollary 4.1 in [17]) we have: $\forall \Phi_h = (\phi_{hij})$ with $\phi_{hij} = \phi_{hji} \in P_m(T)$, $m \geqslant 1$, $\forall b \in B_{k+1}(T)$, $k \geqslant 2$,

$$\int_T \phi_{hij} b_{,ij}\, \mathrm{d}T = \int_T \phi_{hij,ij} b\, \mathrm{d}T + \int_{\partial T} M_n(\Phi_h) \frac{\partial b}{\partial n}\, \mathrm{d}s. \tag{5.34}$$

First of all, we will prove (5.31). In fact, for $k \geqslant 6$, for any nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24), we have (5.25)–(5.26). Then, the equality (5.31) follows from (5.34), (5.25) and (5.26).

Now, we prove (5.32). Since $x_1 \phi^*_{hij} = x_1 \phi^*_{hji} \in P_m(T)$, $\forall i, j = 1, 2$, $m = k + 1$, from (5.34) we have for $k \geqslant 6$:

$$\int_T (\phi^*_{hij} x_1) b_{,ij}\, \mathrm{d}T = \int_T (x_1 \phi^*_{hij})_{,ij} b\, \mathrm{d}T + \int_{\partial T} M_n((x_1 \phi^*_{hij})) \frac{\partial b}{\partial n}\, \mathrm{d}s.$$

Since for $k \geqslant 6$, for nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24), we have

$$M_n((x_1 \phi^*_{hij})) = x_1 \phi^*_{hij} n_i n_j = 0$$

by virtue of (5.25). Hence, for $k \geqslant 6$ for nonzero $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ satisfying (5.21)–(5.24), we have, from (5.26) and (5.23):

$$\int_T (\phi^*_{hij} x_1) b_{,ij}\, \mathrm{d}T = \int_T [(x_1 \phi^*_{h11})_{,11} + (x_1 \phi^*_{h12})_{,12} + (x_1 \phi^*_{h21})_{,21} + (x_1 \phi^*_{h22})_{,22}] b\, \mathrm{d}T$$

$$= \int_T x_1 \phi^*_{hij,ij} b\, \mathrm{d}T + 2 \int_T (\phi^*_{h11,1} + \phi^*_{h21,2}) b\, \mathrm{d}T = 0.$$

Similarly, (5.33) is proved. Thus, we have proved that for $k \geqslant 6$, $\exists$ a *nonzero* $\Phi^*_h = (\phi^*_{hij}) \in \mathbf{V}_T$ such that (5.16)–(5.18) hold, i.e. for $k \geqslant 6$, Σ_T defined by (5.14) is *not* $\mathbf{V}_T$-unisolvent.

Proposition 5.3

For $1 \leqslant k \leqslant 3$, the degrees of freedom ($\Sigma 1$) for tensor-valued functions $\Phi_h = (\phi_{hij}) \in \mathbf{V}_h$ can be defined by the values of

$$\int_{T_i^*} M_n(\Phi_h) q \, \mathrm{d}s, \quad q \in P_k(T_i^*), \quad 1 \leqslant i \leqslant 3;$$

$$\int_{T_i^*} K_n(\Phi_h) q \, \mathrm{d}s, \quad q \in P_{k-1}(T_i^*), \quad 1 \leqslant i \leqslant 3,$$

$$\int_T \phi_{hij}(\lambda_1 \lambda_2 \lambda_3 p)_{,ij} q \, \mathrm{d}T, \quad p \in P_{k-2}(T), \quad q \in P_1(T); \quad T \in T_h. \tag{5.35}$$

Proof. The result follows from Lemma 5.1.

Remark 5.3

From Lemma 5.2, it follows that for $k \geqslant 6$, ($\Sigma 1$) defined in (5.35) do *not* define degrees of freedom for functions $\Phi_h \in \mathbf{V}_h$, but for $k = 4, 5$ the problem is still open, i.e. it is still not known whether ($\Sigma 1$) in (5.35) define degrees of freedom for functions $\Phi_h \in \mathbf{V}_h$ or not.

Corresponding to ($\Sigma 1$) given by (5.35), we define the linear operator $\mathbf{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ [13, 16, 27] as follows: $\forall \Phi \in \mathbf{V}$, $\mathbf{\Pi}_h \Phi \in \mathbf{V}_h$ such that

$$\int_{T^*} M_n(\Phi - \mathbf{\Pi}_h \Phi) q \, \mathrm{d}s = 0, \quad q \in P_k(T^*), \quad T^* \in S_h, \quad 1 \leqslant k \leqslant 3, \tag{5.36}$$

$$\int_{T_i^*} K_n(\Phi - \mathbf{\Pi}_h \Phi) q \, \mathrm{d}s = 0, \quad q \in P_{k-1}(T^*), \quad T^* \in S_h, \quad 1 \leqslant k \leqslant 3, \tag{5.37}$$

$$\int_T \phi_{ij} - (\mathbf{\Pi}_h \Phi)_{ij} (\lambda_1 \lambda_2 \lambda_3 p)_{,ij} q \, \mathrm{d}T = 0, \quad p \in P_{k-2}(T), \quad q \in P_1(T), \quad 1 \leqslant i, j \leqslant 2, \quad 2 \leqslant k \leqslant 3. \tag{5.38}$$

Lemma 5.3 [13, 17]

Let $\mathbf{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ be defined by (5.36)–(5.38). Then

$$b(\Phi - \mathbf{\Pi}_h \Phi, \chi_h) = 0, \quad \forall \chi_h \in W_h, \quad \forall \Phi \in \mathbf{V}. \tag{5.39}$$

Lemma 5.4 [13, 16]

For a regular family $\{T_h\}$ [20] of triangulations of $\bar{\Omega}$ and $\mathbf{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ defined by (5.36)–(5.38), $\exists C^* > 0$ such that

$$\| \mathbf{\Pi}_h \Phi \|_{\mathbf{V}} \leqslant C^* \| \Phi \|_{\mathbf{V}}, \quad \forall \Phi \in \mathbf{V}. \tag{5.40}$$

Moreover, $\forall \Phi \in (H^{k+1}(\Omega))^4 \cap \mathbf{V}$, $\exists C_1^* > 0$ such that

$$\| \Phi - \mathbf{\Pi}_h \Phi \|_{l,\Omega} \leqslant C_1^* h^{k+1-l} | \Phi |_{k+1,\Omega}, \quad 0 \leqslant l \leqslant k+1. \tag{5.41}$$

Proposition 5.4

For the degrees of freedom ($\Sigma 1$) defined in (5.35), the assumption (A3) holds.

Proof. Let $\mathbf{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ be defined by (5.36)–(5.38). Then, from (4.25) and (5.40), we have

$$\sup_{\Phi_h \in \mathbf{V}_h} \frac{|b(\Phi_h, \chi_h)|}{\| \Phi_h \|_{\mathbf{V}}} \geqslant \sup_{\Phi \in \mathbf{V}} \frac{|b(\mathbf{\Pi}_h \Phi, \chi_h)|}{\| \mathbf{\Pi}_h \Phi \|_{\mathbf{V}}} = \sup_{\Phi \in \mathbf{V}} \frac{|b(\Phi, \chi_h)|}{\| \Phi \|_{\mathbf{V}}} \frac{\| \Phi \|_{\mathbf{V}}}{\| \mathbf{\Pi}_h \Phi \|_{\mathbf{V}}} \geqslant \frac{\beta}{C^*} \| \chi_h \|_{0,\Omega},$$

$\forall \chi_h \in W_h$, from which the result follows with $\beta_1 = \beta / C^* > 0$.

Corresponding to ($\Sigma 1$), we constructed $\mathbf{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ to prove that (A3) holds which in turn, assures the existence and uniqueness of the solution of ($\mathrm{Q_h}$). But conversely, we have:

Lemma 5.5 [8]

If (A3) holds, then $\exists$ a linear operator $\underset{\sim}{\Pi}_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ satisfying:

$$b(\Phi - \underset{\sim}{\Pi}_h \Phi, \chi_h) = 0, \quad \forall \Phi \in \mathbf{V}, \quad \forall \chi_h \in \mathbf{W}_h; \quad \|\underset{\sim}{\Pi}_h \Phi\|_{\mathbf{V}} \leqslant \frac{m^*}{\beta_1}\|\Phi\|_{\mathbf{V}}, \quad \forall \Phi \in \mathbf{V}$$

m^* and β_1 being positive constants in (4.23) and (A3) respectively.

6. ERROR ESTIMATES

Define

$$X_h = \{\chi_h : \chi_h \in C^0(\bar{\Omega}), \chi_h|_T \in P_1(T), \forall T \in T_h\} \subset W_h. \tag{6.1}$$

Then, for $k = 1$, we have from (5.3) and (5.6),

$$X_h = W_h. \tag{6.2}$$

Let g_h: $W \to X_h$ be defined by:

$$\forall \chi \in W \hookrightarrow C^0(\bar{\Omega}), \quad g_h\chi \in X_h \quad \text{and} \quad (g_h\chi)(a_{i,T}) = \chi(a_{i,T}), \quad 1 \leqslant i \leqslant 3, \quad \forall T \in T_h. \tag{6.3}$$

Remark 6.1

$\forall \chi \in W$, $g_h\chi$ is the standard Lagrange interpolant of χ at the vertices of triangles $T \in T_h$.

Proposition 6.1

$\forall \Phi_h \in \mathbf{V}_h$, $\forall \chi \in W$,

$$|b(\Phi_h, \chi - g_h\chi)| \leqslant \|\Phi_h\|_{\mathbf{V}} \|\chi - g_h\chi\|_{0,\Omega}, \tag{6.4}$$

where $g_h \in \mathscr{L}(W, X_h)$ is defined by (6.3).

Proof. Since $(\chi - g_h\chi)(a_{i,T}) = 0, \forall i = 1, 2, 3$, $\forall T \in T_h$, the result follows from the application of the Cauchy–Schwarz inequality to (4.21).

Define

$$\mathbf{Z}(f) = \{\Phi : \Phi \in \mathbf{V}, b(\Phi, \chi) = -\langle f, \chi \rangle_{0,\Omega}, \forall \chi \in \mathbf{W}\},$$
$$\mathbf{Z} = \mathbf{Z}(0) = \{\Phi : \Phi \in \mathbf{V}, b(\Phi, \chi) = 0, \forall \chi \in W\}; \tag{6.5}$$
$$\mathbf{Z}_h(f) = \{\Phi_h : \Phi_h \in \mathbf{V}_h, b(\Phi_h, \chi_h) = -\langle f, \chi_h \rangle_{0,\Omega}, \forall \chi_h \in W_h\},$$
$$\mathbf{Z}_h = \mathbf{Z}_h(0) = \{\Phi_h : \Phi_h \in \mathbf{V}_h, b(\Phi_h, \chi_h) = 0, \forall \chi_h \in W_h\}, \tag{6.6}$$

$\mathbf{Z}$ and $\mathbf{Z}_h$ being subspaces of $\mathbf{V}$ and $\mathbf{V}_h$ respectively, although $\mathbf{Z}_h \not\subset \mathbf{Z}$ in general. But we have:

Lemma 6.1 [13, 17]

For $1 \leqslant k \leqslant 3$, let ($\Sigma$1) be the degrees of freedom of $\Phi_h \in \mathbf{V}_h$ defined by (5.35). Then, $\mathbf{Z}_h \subset \mathbf{Z}$.

Theorem 6.1

For $0 < h < 1$, let $\{T_h\}$ be a regular family of triangulations of $\bar{\Omega}$. Then, $\exists$ constants $C_1, C_2 > 0$, independent of h, such that

$$\|\Psi - \Psi_h\|_{0,\Omega} \leqslant C_1 \|\Psi - \Pi_h \Psi\|_{0,\Omega}; \tag{6.7}$$

$$\|\lambda - \lambda_h\|_{0,\Omega} \leqslant C_2(\|\lambda - g_h\lambda\|_{0,\Omega} + \|\Psi - \Psi_h\|_{0,\Omega}), \tag{6.8}$$

where $(\Psi, \lambda) \in \mathbf{V} \times W$, $(\Psi_h, \lambda_h) \in \mathbf{V}_h \times W_h$ are the solutions of the problems (Q) and ($\mathrm{Q_h}$) respectively, $\Pi_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$, $g_h \in \mathscr{L}(W, X_h)$ are defined by (5.36)–(5.38) and (6.1)–(6.3) respectively.

Proof. From Lemma 5.3, $\forall \Phi \in \mathbf{V}$, $b(\Phi - \Pi_h \Phi, \chi_h) = 0$, $\forall \chi_h \in W_h$. Then

$$b(\Psi - \Pi_h \Psi, \chi_h) = 0, \quad \forall \chi_h \in W_h \Rightarrow b(\Pi_h \Psi, \chi_h) = b(\Psi, \chi_h) = -\langle f, \chi_h \rangle, \quad \forall \chi_h \in W_h$$
$$\Rightarrow \Pi_h \Psi \in \mathbf{Z}_h(f) \quad \text{by (6.5).} \tag{6.9}$$

Since $\Psi_h \in \mathbf{Z}_h(f)$, $\forall \Phi_h \in \mathbf{Z}_h(f)$, $\Psi_h - \Phi_h \in \mathbf{Z}_h \Rightarrow \Psi_h - \Phi_h \in \mathbf{Z}$ by virtue of the inclusion $\mathbf{Z}_h \subset \mathbf{Z}$ (Lemma 6.1). Then, $b(\Psi_h - \Phi_h, \chi - \chi_h) = 0$, $\forall \Phi_h \in \mathbf{Z}_h(f)$, $\forall \chi_h \in W_h$. Hence,

$$b(\Psi_h - \Phi_h, \lambda - \lambda_h) = 0, \quad \forall \Phi_h \in \mathbf{Z}_h(f) \Rightarrow b(\Psi_h - \Pi_h \Psi, \lambda - \lambda_h) = 0 \tag{6.10}$$

by (6.9). Again, from (4.26), (5.10) and (6.10) we have

$$A(\Psi - \Psi_h, \Psi_h - \Pi_h \Psi) + b(\Psi_h - \Pi_h \Psi, \lambda - \lambda_h) = A(\Psi - \Psi_h, \Psi_h - \Pi_h \Psi) = 0. \tag{6.11}$$

Then, from (4.22), (4.24) and (6.11), we have

$$\begin{aligned} \alpha_2 \| \Psi_h - \Pi_h \Psi \|_{0,\Omega}^2 &\leqslant A(\Psi_h - \Pi_h \Psi, \Psi_h - \Pi_h \Psi) \\ &= A(\Psi - \Pi_h \Psi, \Psi_h - \Pi_h \Psi) - A(\Psi - \Psi_h, \Psi_h - \Pi_h \Psi) \\ &= A(\Psi - \Pi_h \Psi, \Psi_h - \Pi_h \Psi) \\ &\leqslant M \| \Psi - \Pi_h \Psi \|_{0,\Omega} \| \Psi_h - \Pi_h \Psi \|_{0,\Omega} \end{aligned}$$

$$\Rightarrow \| \Psi_h - \Pi_h \Psi \|_{0,\Omega} \leqslant \frac{M}{\alpha_2} \| \Psi - \Pi_h \Psi \|_{0,\Omega}. \tag{6.12}$$

Now, from (6.12) and the inequality: $\| \Psi - \Psi_h \|_{0,\Omega} \leqslant \| \Psi - \Pi_h \Psi \|_{0,\Omega} + \| \Psi_h - \Pi_h \Psi \|_{0,\Omega}$, the result (6.7) follows with $C_1 = (1 + M/\alpha_2) > 0$. Now, we prove (6.8). From (4.26) and (5.10) we have: $\forall \Phi_h \in \mathbf{V}_h$

$$b(\Phi_h, \lambda_h - \lambda) = A(\Psi - \Psi_h, \Phi_h) \Rightarrow b(\Phi_h, \lambda_h - g_h \lambda) + b(\Phi_h, g_h \lambda - \lambda) = A(\Psi - \Psi_h, \Phi_h), \quad \forall \Phi_h \in \mathbf{V}_h, \tag{6.13}$$

where $g_h \in \mathscr{L}(W, X_h)$ is defined by (6.1)–(6.3). Then, since for $\Pi_h \in \mathscr{L}(\mathbf{V}, \mathbf{V}_h)$ defined by (5.36)–(5.38) corresponding to (Σ1) in (5.35), the (A3) holds, we have from (6.13):

$$\begin{aligned} \beta_1 \| \lambda_h - g_h \lambda \|_{0,\Omega} &\leqslant \sup_{\Phi_h \in \mathbf{V}_h} \frac{| b(\Phi_h, \lambda_h - g_h \lambda) |}{\| \Phi_h \|_{\mathbf{V}}} \\ &\leqslant \sup_{\Phi_h \in \mathbf{V}_h} \frac{| A(\Psi - \Psi_h, \Phi_h) |}{\| \Phi_h \|_{\mathbf{V}}} + \sup_{\Phi_h \in \mathbf{V}_h} \frac{| b(\Phi_h, \lambda - g_h \lambda) |}{\| \Phi_h \|_{\mathbf{V}}} \\ &\leqslant M \| \Psi - \Psi_h \|_{0,\Omega} + \| \lambda - g_h \lambda \|_{0,\Omega} \quad \text{(by virtue of (6.4))} \end{aligned}$$

$$\Rightarrow \| \lambda_h - g_h \lambda \|_{0,\Omega} \leqslant (M/\beta_1) \| \Psi - \Psi_h \|_{0,\Omega} + \frac{1}{\beta_1} \| \lambda - g_h \lambda \|_{0,\Omega}. \tag{6.14}$$

Now, from (6.14) and the inequality: $\| \lambda - \lambda_h \|_{0,\Omega} \leqslant \| \lambda - g_h \lambda \|_{0,\Omega} + \| \lambda_h - g_h \lambda \|_{0,\Omega}$, the result (6.8) follows with $C_2 = \max\{1 + 1/\beta_1, M/\beta_1\}$.

The final result is given by:

Theorem 6.2

For $0 < h < 1$, let $\mathbf{V}_h$, W_h be the finite dimensional vector spaces defined by (5.5) and (5.6) corresponding to a regular family $\{T_h\}$ of triangulations of $\bar{\Omega}$ and the degrees of freedom (Σ1) defined by (5.35). If the solution $u \in H_0^2(\Omega)$ of the problem ($\mathrm{P_G}$) belongs to $H^{k+3}(\Omega) \cap H_0^2(\Omega)$, $1 \leqslant k \leqslant 3$, such that $a_{ijml} u_{,ml} \in H^{k+1}(\Omega)$, $\forall i, j = 1, 2$, then $\exists$ constants C_3 and $C_4 > 0$, independent of h, such that

$$\| \Psi - \Psi_h \|_{0,\Omega} \leqslant C_3 h^{k+1} | \Psi |_{k+1,\Omega} \quad (1 \leqslant k \leqslant 3); \tag{6.15}$$

$$\| \lambda - \lambda_h \|_{0,\Omega} \leqslant C_4 h^2 (| \lambda |_{2,\Omega} + | \Psi |_{k+1,\Omega}) \quad (1 \leqslant k \leqslant 3), \tag{6.16}$$

where $(\Psi, \lambda) \in \mathbf{V} \times W$ is the solution of the problem (Q) with $\lambda = u$, $\Psi = (\psi_{ij})$, $1 \leqslant i, j \leqslant 2$, $\psi_{ij} = a_{ijml} u_{,ml}$, $\forall i, j = 1, 2$; $(\Psi_h, \lambda_h) \in \mathbf{V}_h \times W_h$ is the equilibrium finite element solution of ($\mathrm{Q_h}$).

Proof. For $1 \leqslant k \leqslant 3$ let $u \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$ be the solution of the problem ($\mathrm{P_G}$) with $a_{ijml} u_{,ml} \in H^{k+1}(\Omega)$, $\forall i, j = 1, 2$. Then, the solution $(\Psi, \lambda) \in \mathbf{V} \times W$ of the problem (Q) will have the regularity defined by:

$$\lambda = u \in H^{k+3}(\Omega) \cap H_0^2(\Omega), \quad \Psi = (\psi_{ij}) \in (H^{k+1}(\Omega))^4, \quad 1 \leqslant k \leqslant 3, \tag{6.17}$$

since $\forall i, j = 1, 2$, $\psi_{ij} = a_{ijml}\lambda_{,ml} = a_{ijml}u_{,ml} \in H^{k+1}(\Omega)$ by virtue of (4.28).

Then, from (5.41), we have

$$\| \Psi - \Pi_h \Psi \|_{0,\Omega} \leqslant C_1^* h^{k+1} | \Psi |_{k+1,\Omega}, \quad 1 \leqslant k \leqslant 3, \tag{6.18}$$

and the result (6.15) follows from (6.7) with $C_3 = C_1 C_1^* > 0$. Now, we prove (6.16).

Since $(g_h\lambda)|_T \in P_1(T)$, $\forall T \in T_h$, $\lambda \in H^{k+3}(\Omega) \cap H_0^2(\Omega)$, $1 \leqslant k \leqslant 3$, we have the classical result [20]:

$$\| \lambda - g_h \lambda \|_{0,\Omega} \leqslant C_5 h^2 |\lambda|_{2,\Omega}, \quad C_5 > 0. \tag{6.19}$$

Then from (6.8), (6.19) and (6.15), the result (6.16) follows with

$$C_4 = \max\{C_2C_5, C_2C_3\} > 0.$$

7. NUMERICAL RESULTS

The following problems have been considered for numerical computations.

I. Biharmonic (Stokes) problem

$$\Lambda u \equiv \Delta\Delta u = f \quad \text{in} \quad \Omega; \quad u|_\Gamma = \left.\frac{\partial u}{\partial n}\right|_\Gamma = 0;$$

data: $\Omega = (0, 1) \times (0, 1)$, $\bar{\Omega} = \Omega \cup \Gamma = [0, 1] \times [0, 1]$,

$$f(x_1, x_2) = 24(x_1^2(x_1 - 1)^2 + x_2^2(x_2 - 1)^2) + 8(6x_1^2 - 6x_1 + 1)(6x_2^2 - 6x_2 + 1), \quad \forall (x_1, x_2) \in \Omega.$$

The *exact solution* [16, 17] of the problem is

$$u(x_1, x_2) = x_1^2 x_2^2 (x_1 - 1)^2 (x_2 - 1)^2. \tag{7.1}$$

The results of the numerical experiment are given in Table 1.

II. Bending problems of clamped elastic plates

(i) *Isotropic case* [see (4.41), (4.39) and (4.40)]: thickness $h \equiv$ const.

$$\Lambda u \equiv D\Delta\Delta u = f \quad \text{in} \quad \Omega; \quad u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0;$$

data: $\Omega = (-1/2, 1/2) \times (-3/4, 3/4)$, $\bar{\Omega} = [-1/2, 1/2] \times [-3/4, 3/4]$, $f = q =$ const., $\nu = 0.3$, $D = Eh^3/(12(1 - \nu^2))$.

The *Timoshenko solution* [25] of the problem gives

$$u(0,0) = \alpha_T q/D; \quad \psi_{ii}(0,0) = \beta_{iT} q \quad (i = 1,2); \quad \psi_{11}(1/2, 0) = \gamma_{1T} q; \quad \psi_{22}(0, 3/4) = \gamma_{2T} q, \tag{7.2}$$

where the values of α_T, β_{iT}, γ_{iT} $(i = 1, 2)$ are those given along with the results of numerical experiment for the isotropic case in Table 2.

(ii) *Orthotropic case* [see (4.39) and (4.40)]: thickness $h =$ const.,

$$\Lambda u \equiv D_1 u_{,1111} + 2H u_{,1122} + D_2 u_{,2222} = f \quad \text{in} \quad \Omega; \quad u|_\Gamma = (\partial u/\partial n)|_\Gamma = 0;$$

data: $\Omega = (-1/2, 1/2) \times (-3/4, 3/4)$, $\bar{\Omega} = [-1/2, 1/2] \times [-3/4, 3/4]$, $f = q =$ const., $h = 0.01$, $\nu_2 = 0.07$, $E_1 = 0.21 \times 10^6$, $E_2 = 0.16 \times 10^6$, $G = 0.84126 \times 10^5$ (all are to be taken in proper units of measurement).

Table 1. Biharmonic (Stokes) problem. $\psi_{hii} = \kappa_{hii} = u_{h,ii}$ $(i = 1, 2)$, $\omega_h = -(\kappa_{h11} + \kappa_{h22})$; $k = 1$

ND1 = ND2	No. of unknowns†	$u_h(1/2, 1/2)$	$\kappa_{h11}(1/2, 1/2)$	$\kappa_{h11}(0, 1/2)$	$\omega_h(1/2, 1/2)$
2	25	0.005327	−0.07873	0.07473	0.1575
4	129	0.003662	−0.06652	0.10991	0.1330
6	313	0.003722	−0.06427	0.11846	0.1285
8	577	0.003785	−0.06347	0.12136	0.1269
Exact solution (7.1)		0.003906	−0.0625	0.125	0.125

†Can be reduced using symmetry of the problem and considering one fourth of $\bar{\Omega}$.

Table 2. Clamped isotropic plate problem. $u_h(0,0)=\alpha_h^I q/D$, $\psi_{hii}(0,0)=\beta_{ih}^I q$ $(i=1,2)$, $\psi_{h11}(1/2,0)=\gamma_{1h}^I q$, $\psi_{h22}(0,3/4)=\gamma_{2h}^I q$, $k=1$

ND1	ND2	No. of unknowns†	α_h^I	β_{1h}^I	β_{2h}^I	γ_{1h}^I	γ_{2h}^I
4	6	201	0.002256	−0.04181	−0.02230	0.06950	0.04894
8	12	881	0.002200	−0.03799	−0.02077	0.07407	0.05504
12	18	2041	0.002197	−0.03730	−0.02049	0.07494	0.05613
Timoshenko solution (7.2)			0.00220 (α_T)	−0.0368 (β_{1T})	−0.0203 (β_{2T})	0.0757 (γ_{1T})	0.0570 (γ_{2T})

†See note to Table 1.

Table 3. Clamped orthotropic plate problem. $u_h(0,0)=\alpha_h^0 q$, $\psi_{hii}(0,0)=\beta_{ih}^0 q$ $(i=1,2)$, $\psi_{h11}(1/2,0)=\gamma_{1h}^0 q$, $\psi_{h22}(0,3/4)=\gamma_{2h}^0 q$, $k=1$

ND1	ND2	No. of unknowns†	α_h^0	β_{1h}^0	β_{2h}^0	γ_{1h}^0	γ_{2h}^0
4	6	201	0.1342	−0.04122	−0.01072	0.07197	0.04224
8	12	881	0.1312	−0.03766	−0.01010	0.07661	0.04772
12	18	2041	0.1311	−0.03699	−0.01000	0.07750	0.04886
Szilard's results (7.3)			0.1416 (α_s)	−0.04115 (β_{1s})	−0.01631 (β_{2s})	0.07981 (γ_{1s})	0.02702 (γ_{2s})

†See note to Table 1.

Numerical values in Tables 1–3 are to be understood in proper units of measurement.

The *Szilard's solution* [24] of the problem gives:

$$u(0,0)=\alpha_s q;\quad \psi_{ii}(0,0)=\beta_{is}q\quad (i=1,2);\quad \psi_{11}(1/2,0)=\gamma_{1s}q;\quad \psi_{22}(0,3/4)=\gamma_{2s}q \tag{7.3}$$

where the values of α_s, β_{is}, γ_{is} $(i=1,2)$ are those given along with the results of the numerical experiments for the orthotropic case in Table 3.

Remark 7.1

The results of Szilard (7.3) given in Table 3 are themselves *not* accurate, rather *very crude approximations* [24]. But these have been included here just for the sake of effecting some comparison of the results of the numerical experiment for the orthotropic case.

Remark 7.2

In all the numerical experiments, the following strategies have been adopted for the sake of simplicity and convenience in computation.

(i) $\bar{\Omega}$ has been triangulated into isosceles triangles as shown in Figs 1 and 2.

(ii) A new discrete probem (Q_h^*) equivalent to the discrete problem (Q_h) in (5.10) and (5.11) in certain sense has been constructed by relaxing the constraints of "continuity" of M_n and K_n across interelement boundaries of the triangulation T_h in the definition of the admissible space $\mathbf{V}_h$ (5.5) with the help of suitable Lagrange multipliers, the space W_h (5.6) being the same for (Q_h^*), and this new discrete problem (Q_h^*) has been used for computational purpose. All the details of the

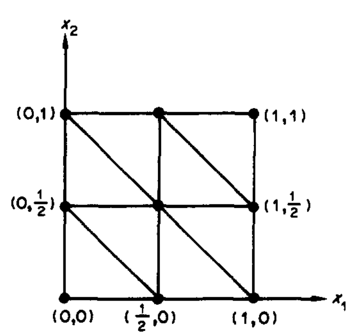

Fig. 1. Biharmonic case. ND1 = No. of subdivisions in x_1 direction; ND2 = No. of subdivisions in x_2 direction. Here ND1 = ND2 = 2; No. of triangles = 8.

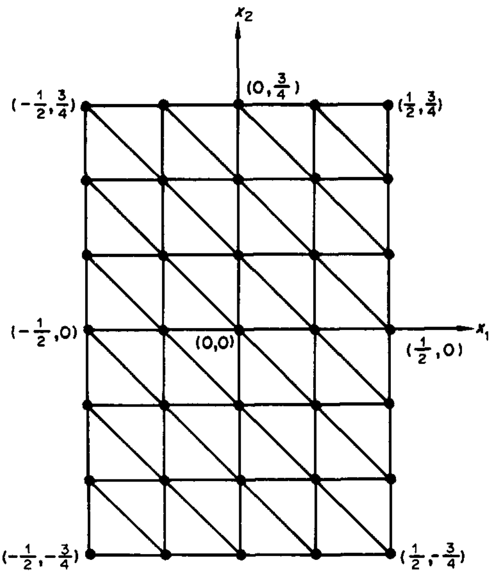

Fig. 2. Isotropic and orthotropic cases. Here ND1 (see Fig. 1 caption) = 4, ND2 = 6; No. of triangles = 48.

construction of this new discrete problem (Q_h^*), the corresponding new admissible spaces, the reduction of (Q_h^*) to matrix form and finally, the solution procedures involved in the numerical computations can be found in [17], but the details of computer implementations of this new scheme will be dealt with in a future publication of the authors.

(iii) Only the points of $\bar{\Omega}$ at which u_h and ψ_{hii} $(i = 1, 2)$ attain optimal values have been considered in Tables 1–3.

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REFERENCES

1. S. Balasundaram and P. K. Bhattacharyya, A mixed finite element method for the Dirichlet problem of fourth order elliptic operators with variable coefficients. *4th Int. Symp. Finite Element Methods in Flow Problems*, Tokyo, July (1982).
2. S. Balasundaram and P. K. Bhattacharyya, A mixed finite element method for fourth order elliptic partial differential equations. *Z. angew. Math. Mech.* Vol. 66, No. 10, 489–499 (1986).
3. S. Balasundaram and P. K. Bhattacharyya, A mixed finite element method for the Dirichlet problem of fourth order elliptic equations with variable coefficients. *Int. Conf. Finite Element Methods*, Shanghai, August (1982).
4. S. Balasundaram and P. K. Bhattacharyya, A mixed finite element method for fourth order elliptic equations with vairable coefficients. *Comput. Math. Applic.* **10**, 245–256 (1984).
5. S. Balasundaram and P. K. Bhattacharyya, A mixed finite element method for fourth order elliptic equations with variable coefficients. Unpublished.
6. I. Babuska, J. Osborn and J. Pitkaranta, Analysis of mixed methods using mesh dependent norms. *Math. Comput.* **35**, 1039–1062 (1980).
7. F. Brezzi and P. A. Raviart, Mixed finite element methods for fourth order elliptic equations. *Topics in Numerical Analysis*, Vol. 3 (Edited by J. Miller). Academic Press, New York (1978).
8. R. S. Falk and J. E. Osborn, Error estimates for mixed methods. *R.A.I.R.O. Analyse numérique* **14**(3), 249–277 (1980).
9. K. Hellan, Analysis of elastic plates in flexure by a simplified finite element method. *Acta polytech. scand. Civil Engng. Ser.* **46** (1967).
10. L. Hermann, Finite element bending analysis for plates. *J. Engng. Mech. Div. ASCE EM5* **93**, 49–83 (1967).
11. C. Johnson, On the convergence of a mixed finite element method for plate bending problems. *Numer. Math.* **21**, 43–62 (1973).
12. J. T. Oden and G. F. Carey, *Finite Elements—Mathematical Aspects*, Vol. 4. The Texas Finite Element Series, Prentice Hall, New Jersey (1983).
13. P. A. Raviart, Mixed and equilibrium methods for 4th order problems. Rapport de Recherche, Laboratoire d'Analyse Numérique, Université de Paris VI (1976).
14. B. Fraeijs de Veubeke, Displacement and equilibrium models in the finite element method. In *Stress Analysis* (Edited by O. C. Zienkiewicz and G. S. Holister), Chapter 9. Wiley, New York (1974).
15. B. Fraeijs de Veubeke and G. Sander, An equilibrium model for plate bending. *Int. J. Solids Struct.* **4**, 447–468 (1968).

16. F. Brezzi, L. D. Marini, A. Quarteroni and P. A. Raviart, On an equilibrium finite element method for plate bending problems. Publ. No. 169, L.A.N. of C.N.R., Pavia (1979).
17. P. K. Bhattacharyya and S. Gopalsamy, An equilibrium finite element method for fourth order elliptic equations with variable coefficients. Research Report No. 315, Institut National de Recherche en Informatique et en Automatique, Roequencourt, France (1984).
18. R. A. Adams, *Sobolev Spaces*. Academic Press, New York (1975).
19. J. L. Lions, *Problèmes aux limites dans les èquations derivèes partielles*. Les Presses de l'Université de Montréal, Montréal (1965).
20. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*. North Holland, Amsterdam (1977).
21. S. Balasundaram and P. K. Bhattacharyya, On existence of solution of the Dirichlet problem of fourth order partial differential equations with variable coefficients. *Q. Jl appl. Math.* **39**, 311–317 (1983).
22. V. A. Kondratev, Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Mosk. Mat. Obsc.* **16**, 209–292 (1967).
23. S. G. Lekhnitskii, *Anisotropic Plates*. Gordon & Breach, New York (1968).
24. R. Szilard, *Theory and Analaysis of Plates*. Prentice Hall, New Jersey (1974).
25. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*. McGraw-Hill, New York (1976).
26. O. C. Zienkiewicz, *The Finite Element Method*. McGraw-Hill, New York (1977).
27. M. Fortin, Analysis of the convergence of mixed finite element methods. *R.A.I.R.O.* **11**, 341–354 (1977).